

- Recap: ① Some basic motivations of the linear model  
 ② basic settings of types of linear models  
 ③ Design of experiment.

### One way Anova:

Eg:  $y_{ij}$  is the  $j$ th measurement of nitrogen conc. in the soil that received treatment  $i$ .

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

### ② standard linear model form

$$Y = X\beta + \epsilon$$

$$i=1, \dots, a, \quad j=1, \dots, n$$

$$y_{11} = \mu + \alpha_1 + \epsilon_{11}$$

$$y_{12} = \mu + \cancel{\alpha_1} + \alpha_1 + \epsilon_{12}$$

$$\vdots$$

$$y_{1n} = \mu + \alpha_1 + \epsilon_{1n}$$

$$y_{21} = \mu + \alpha_2 + \epsilon_{21}$$

$$\vdots$$

$$y_{2n} = \mu + \alpha_2 + \epsilon_{2n}$$

$$\vdots$$

$$y_{a1} = \mu + \alpha_a + \epsilon_{a1}$$

$$\vdots$$

$$y_{an} = \mu + \alpha_a + \epsilon_{an}$$

$$\left( \begin{array}{c} y_{11} \\ \vdots \\ y_{an} \end{array} \right) = \left[ \begin{array}{cccc} 1_n & 1_n & \cdots & 0_n \\ \vdots & 0_n & & \vdots \\ 1_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & & 1_n \end{array} \right] \left( \begin{array}{c} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{array} \right) + \left( \begin{array}{c} \epsilon_{11} \\ \vdots \\ \epsilon_{an} \end{array} \right)$$

Let  $n=2$ ,  $a=3$

$$\begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{22} \\ y_{31} \\ y_{32} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \begin{pmatrix} \epsilon_{11} \\ \vdots \\ \epsilon_{32} \end{pmatrix}$$

The above is the linear model representation of the one way anova,

$$X = \begin{bmatrix} \underline{1}_n & \underline{1}_n & \underline{0}_n & \cdots & \underline{0}_n \\ \underline{1}_n & \underline{0}_n & \underline{1}_n & \underline{0}_n & \cdots & \underline{0}_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \underline{1}_n & \cdots & \cdots & \ddots & \cdots & \underline{1}_n \end{bmatrix} \quad \underline{y} = \begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{31} \\ y_{32} \end{pmatrix}$$

Two way Anova model

where one uses two different treatments at multiple levels.

Let, the ~~the~~ first treatment be used at  $a$  different levels and the second treatment is used at  $b$  different levels.

$$y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}, \quad i=1, \dots, a; j=1, \dots, b; k=1, \dots, n.$$

$$\begin{pmatrix} y_{111} \\ \vdots \\ y_{11n} \\ \vdots \\ y_{ab1} \\ \vdots \\ y_{abn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_a \\ \beta_1 \\ \vdots \\ \beta_b \end{pmatrix} + \begin{pmatrix} e_{111} \\ \vdots \\ e_{11n} \\ \vdots \\ e_{ab1} \\ \vdots \\ e_{abn} \end{pmatrix}$$

$$y = X \beta + \epsilon$$

$$X = \begin{bmatrix} 1_n & 1_n & 0_n & \cdots & 0_n & 1_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & 1_n & \cdots & 0_n & 0_n & 1_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_n & 0_n & 1_n & \cdots & 0_n & 1_n & 0_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_n & 1_n & 1_n & \cdots & 1_n & 0_n & 1_n & \cdots & 1_n \end{bmatrix}$$

$$\begin{pmatrix} 1_n & 1_n \\ 1_n & 1_n \\ 1_n & 1_n \\ 1_n & 1_n \\ 1_n & 1_n \\ 1_n & 1_n \\ 1_n & 1_n \\ 1_n & 1_n \\ 1_n & 1_n \end{pmatrix}$$

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$$A, B \text{ are two matrices}$$

$$A = ((a_{ij}))_{i=1, j=1}^{n, m} \quad B = ((b_{ij}))_{i=1, j=1}^{k, l}$$

$$A \otimes B = ((a_{ij}B)) = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ \vdots & \vdots & & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix}$$

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

If there is an interaction between the factors (variables) of the treatments, i.e. their effects are not additive

$$y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk}$$

$$i=1, \dots, a; j=1, \dots, b; k=1, \dots, n$$

$n$  is the number of replications of any pair of factors from treatment 1 & 2.

In order to estimate  $\delta_{ij}$ ,  $n > 1$

### ANCOVA

Recall the nitrogen concentration example. Suppose that the amount of nitrogen measured in each plot of ground prior to the application of any treatment is also added to the model.

Suppose, the  $k$ th plot where  $i$ th factor of treatment A and  $j$ th factor of treatment B are added had a nitrogen conc. of  $x_{ijk}$ .

$$y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + x_{ijk} \gamma + \epsilon_{ijk}$$

Here the parameters are  $\mu, \alpha_1, \dots, \alpha_a, \beta_1, \dots, \beta_b, \delta_{ij}, \dots$

$\dots, \gamma_{ab}, \frac{g}{2}.$

All these one-way, two-way Anova can be written  $\underline{y} = \underline{x}\underline{\beta} + \underline{\epsilon}$

Least square estimation of the parameter  $\underline{\beta}$

Def: Length of vector  $\underline{x} \in \mathbb{R}^n$  is given by

$$\|\underline{x}\| = \sqrt{\underline{x}^T \underline{x}} = \left( \sum_{i=1}^n x_i^2 \right)^{1/2}$$

Distance between two vectors  $\underline{x}, \underline{y} \in \mathbb{R}^n$  is given by

$$\|\underline{x} - \underline{y}\| = \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2}.$$

Least square estimate of  $\underline{\beta}$  is obtained by

minimizing  $\mathcal{Q}(\underline{\beta}) = \|\underline{y} - \underline{x}\underline{\beta}\|^2 = (\underline{y} - \underline{x}\underline{\beta})^T (\underline{y} - \underline{x}\underline{\beta})$   
w.r.t.  $\underline{\beta}$ .

Derivative w.r.t. a vector

Let  $\underline{a}, \underline{b}$  be  $p \times 1$  vectors and  $\underline{A}$  be a  $p \times p$  matrix  
then,

$$\textcircled{1} \quad \frac{\partial \underline{b}^T \underline{a}}{\partial \underline{b}} = \frac{\partial \underline{a}^T \underline{b}}{\partial \underline{b}} = \underline{a}$$

$$\textcircled{2} \quad \frac{\partial \underline{b}^T \underline{A} \underline{b}}{\partial \underline{b}} = (\underline{A} + \underline{A}^T) \underline{b}$$

$$\mathcal{Q}(\underline{\beta}) = \underline{y}^T \underline{y} - \underline{y}^T \underline{x} \underline{\beta} - \underline{\beta}^T \underline{x}^T \underline{y} + \underline{\beta}^T \underline{x}^T \underline{x} \underline{\beta}$$

$$\begin{aligned} \frac{\partial \mathcal{Q}(\underline{\beta})}{\partial \underline{\beta}} &= 0 - (\underline{y}^T \underline{x})^T - \underline{x}^T \underline{y} + (\underline{x}^T \underline{x} + (\underline{x}^T \underline{x})^T) \underline{\beta} \\ &= -2 \underline{x}^T \underline{y} + 2 \underline{x}^T \underline{x} \underline{\beta} \end{aligned}$$

$$\frac{\partial \underline{Q}(\underline{\beta})}{\partial \underline{\beta}} = \underline{0} \Rightarrow \underline{x}^T \underline{x} \underline{\beta} = \underline{x}^T \underline{y} \quad (\text{Normal Equations}) \\ (\text{NE})$$

② Goal: Find  $\underline{\beta}$ .

The solution is not always unique. We will characterize all these solutions.

Example:  $y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \quad i=1, \dots, n$ .

$$\underline{x} = \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \underline{y} = \underline{x} \underline{\beta} + \underline{\epsilon}$$

$$(\underline{x}' \underline{x}) \underline{\beta} = \underline{x}' \underline{y} \quad \cdot \quad \underline{x}' \underline{x} = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \\ = \begin{pmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix}$$

$$\underline{x}' \underline{y} = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix}$$

$$\begin{pmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{pmatrix} \quad (\text{NES})$$

Example: One way Anova with the same number of replications for each treatment

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \quad i=1 \dots a, \quad j=1 \dots n.$$

$$\underline{X} = \begin{bmatrix} \underline{1}_n & \underline{1}_n & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \underline{1}_n \\ \underline{1}_n & 0 & \dots & \underline{1}_n \end{bmatrix} \quad \underline{y} = \begin{pmatrix} y_{11} \\ \vdots \\ y_{an} \end{pmatrix}$$

$$\underline{X}'\underline{X} = \begin{bmatrix} \underline{1}_n & \dots & \dots & \underline{1}_n \\ \underline{1}_n & 0_n & \dots & 0_n \\ 0_n & \underline{1}_n & 0_n & \dots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_n & \dots & \dots & \underline{1}_n \end{bmatrix} \begin{bmatrix} \underline{1}_n & \underline{1}_n & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \underline{1}_n \end{bmatrix}$$

$$= \begin{bmatrix} na & n & n & \dots & n \\ n & n & 0 & \dots & 0 \\ n & 0 & n & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 0 & \dots & \dots & n \end{bmatrix}$$

$$\underline{X}'\underline{y} = \begin{bmatrix} \underline{1}_n & \dots & \dots & \underline{1}_n \\ \underline{1}_n & 0_n & \dots & 0_n \\ \vdots & \vdots & \vdots & \vdots \\ 0_n & \dots & \dots & \underline{1}_n \end{bmatrix} \begin{pmatrix} y_{11} \\ \vdots \\ y_{an} \end{pmatrix} = \left( \begin{array}{c} \sum_{j=1}^n y_{1j} \\ \vdots \\ \sum_{j=1}^n y_{aj} \end{array} \right)$$

$$\begin{bmatrix} na & n & n & \dots & n \\ n & n & 0 & \dots & 0 \\ \vdots & \vdots & 0 & n & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots \\ n & 0 & \dots & \dots & n \end{bmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{pmatrix} = \left( \begin{array}{c} \sum_{i=1}^a \sum_{j=1}^n y_{ij} \\ \vdots \\ \sum_{j=1}^n y_{aj} \end{array} \right) \quad (\text{NEs})$$

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Normal equations are of the form

$$\underline{x}' \underline{x} \oplus \underline{\beta} = \underline{x}' \underline{y}$$

or equivalently it is of the form

$$\underline{A} \oplus \underline{\beta} = \underline{c} \quad \text{where } \underline{A} = \underline{x}' \underline{x}, \underline{c} = \underline{x}' \underline{y}$$

it has a unique solution if  $\underline{A}^{-1}$  exists.

In that case  $\underline{A}^{-1} \underline{c}$  is the unique solution of  $\underline{\beta}$ .  
If  $\underline{A}$  is ~~not~~ singular, then ~~this~~ this system of  
equations has infinitely many solutions.

### Vector space

A set  $S \subset \mathbb{R}^n$  is a vector space if for any  
 $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$  and scalars  $\alpha, \beta$ , operations of  
vector addition and scalar multiplications  
are defined such that

1.  $(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$
2.  $\underline{x} + \underline{y} = \underline{y} + \underline{x}$
3. There exists a vector  $\underline{0} \in S$  s.t.  
 $\underline{x} + \underline{0} = \underline{x} = \underline{0} + \underline{x}$  for any  $\underline{x} \in S$
4. For any  $\underline{x} \in S$ , there exists  $\underline{y} = \underline{0} - \underline{x}$   
such that  $\underline{x} + \underline{y} = \underline{0} = \underline{y} + \underline{x}$
5.  $\alpha(\underline{x} + \underline{y}) = \alpha \underline{x} + \alpha \underline{y}$
6.  $(\alpha + \beta) \underline{x} = \alpha \underline{x} + \beta \underline{x}$
7.  $(\alpha \beta) \underline{x} = \alpha(\beta \underline{x})$
8. There exists a scalar  $\frac{1}{\alpha}$  s.t.  $\frac{1}{\alpha} \underline{x} = \underline{x}$  (Typically,  $\frac{1}{\alpha} = 1$ )

Typically,  $S \subset \mathbb{R}^n$  is a vector space  
 Then  $\underline{x}, \underline{y} \in S \Rightarrow \alpha \underline{x} + \beta \underline{y} \in S$   
 and  $\underline{0} \in S$ .

Ex:  $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

$(x_1, y_1, z_1)\alpha + \beta(x_2, y_2, z_2) \in \mathbb{R}^3$  if  $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$

$$S = \{(x, y, z) \mid x, y \in \mathbb{R}, z \geq 0\}$$

~~(1, 2, 3)~~,  $(1, 2, 3), (3, 5, 8) \in S$

$$-1(1, 2, 3) - 2(3, 5, 8) = (-7, -12, -19) \notin S$$

Subspace:

Let  $S$  be a vector space and  $M$  be a set with  $M \subset S$  and  $M$  is also a vector space.  
 Then  $M$  is called a subspace.

Ex:  $S = \mathbb{R}^3, M = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$

In  $\mathbb{R}^3$ ,  $M$  (which is the  $x$ - $y$  plane is a subspace).

Note: Let  $S$  be a vector space and let  $\underline{x}_1, \dots, \underline{x}_k \in S$ , the set of all linear combinations of  $\underline{x}_1, \dots, \underline{x}_k$  be denoted by  $M$  and is given by

$$M = \{\underline{y} : \underline{y} = \sum_{j=1}^k c_j \underline{x}_j, c_1, \dots, c_k \text{ are coefficients}\}.$$

Check that  $M$  is a subspace of  $S$ .

Def: (Spanning) The set of all linear combinations of  $\underline{x}_1, \dots, \underline{x}_n \in S$  is called the space spanned by  ~~$\underline{x}_1, \dots, \underline{x}_n$~~ .

In the previous example  $M$  is the space spanned by  ~~$\underline{x}_1, \dots, \underline{x}_n$~~ .

Ex:  $S = \mathbb{R}^3$ ,  $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ,  $\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{aligned} M &= \left\{ \underline{y} : \underline{y} = c_1 \underline{e}_1 + c_2 \underline{e}_2 + c_3 \underline{e}_3 \right\} \\ &= \left\{ \underline{y} : \underline{y} = \begin{pmatrix} c_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} \right\} \\ &= \left\{ \underline{y} : \underline{y} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\} \end{aligned}$$

if one ~~one~~ vanishes  $c_1, c_2, c_3$  in  $\mathbb{R}$  one can ~~spanned~~ represent any point  $\mathbb{R}^3$  in this way. Thus, the set, by  $\underline{e}_1, \underline{e}_2, \underline{e}_3$  ~~spans~~ is  $\mathbb{R}^3$ .